Rossmoyne Senior High School

Semester One Examination, 2022

Question/Answer booklet

MATHEMATICS METHODS UNIT 3

SOLUTIONS

Section One: Calculator-free

WA student number: In figures

In words

Time allowed for this section

Reading time before commencing work: five minutes
Working time: fifty minutes

Number of additional answer booklets used (if applicable):

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer booklet
Formula sheet

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	7	7	50	55	35
Section Two: Calculator-assumed	12	12	100	95	65
				Total	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet preferably using a blue/black pen. Do not use erasable or gel pens.

3. You must be careful to confine your answers to the specific question asked and to follow any instructions that are specific to a particular question.

4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

SN085-195-3

Section One: Calculator-free **35% (55 Marks)**

This section has **seven** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 50 minutes.

Question 1 **(6 marks)**

(a) Determine $f'(-2)$ when $f(x) = 2(3x+5)^3$. (3 marks)

Solution
$f'(x) = 2(3)(3)(3x+5)^2$ $= 18(3x+5)^2$ $f'(-2) = 18(-1)^2$ $= 18$
Specific behaviours
✓ recognises the need to use chain rule ✓ obtains correct derivative ✓ obtains correct value

(b) Determine $g(2)$ when $g'(x) = 12e^{3x-3}$ and $g(1) = 7$. (3 marks)

Solution
$g(2) = g(1) + \int_1^2 g'(x)\,dx$ $= 7 + \int_1^2 12e^{3x-3}\,dx$ $= 7 + [4e^{3x-3}]_1^2$ $= 7 + 4e^3 - 4e^0$ $= 3 + 4e^3$
Specific behaviours
✓ indicates total change is integral of rate of change ✓ obtains correct antiderivative ✓ obtains correct value

Solution
$g(x) = \int g'(x)\,dx$ $= \int 12e^{3x-3}\,dx$ $= 4e^{3x-3} + c$ When $g(1) = 7$ $c = 3$ $g(x) = 4e^{3x-3} + 3$ $g(2) = 4e^3 + 3$
Specific behaviours
✓ antidifferentiates to find $g(x)$ ✓ obtains correct antiderivative ✓ obtains correct value

Question 2 **(7 marks)**

Let $f(x) = 15 - 4x - 6x^2 - 4x^3 - x^4$.

(a) The curve $y = f(x)$ cuts the horizontal axis at $x = 1$. State, with reasons, whether the function is increasing, decreasing or neither at this point. (2 marks)

Solution
$f'(x) = -4 - 12x - 12x^2 - 4x^3, \quad f'(1) = -4 - 12 - 12 - 4 = -32$ Accept: $f'(1) = -ve$ Since the gradient at this point is negative, then the function is decreasing.
Specific behaviours
✓ indicates that $f'(1) < 0$ ✓ uses sign of derivative to deduce function is decreasing

(b) Determine $f''(0)$ and use this value to describe the concavity of the curve $y = f(x)$ where it crosses the vertical axis. (2 marks)

Solution
$f''(x) = -12 - 24x - 12x^2, \quad f''(0) = -12$ The curve is concave down at this point.
Specific behaviours
✓ correctly evaluates $f''(0)$, no mark if $f(0) = -ve$ ✓ states concavity

(c) Does the curve $y = f(x)$ have any points of inflection? If it does, determine the coordinates of their location. If not, justify your answer. (3 marks)

<table>
<tr><th colspan="2">Solution</th></tr>
<tr><td colspan="2">No, the curve does not have any points of inflection.

$f''(x) = -12(x^2 + 2x + 1) = -12(x + 1)^2, \quad f''(x) = 0 \Rightarrow x = -1$

Possible point of inflection at $x = -1$, so test for inflection:</td></tr>
<tr><td>$f''(-1.1) = 12(-1.1 + 1)^2 > 0$
$f''(-0.9) = 12(-0.9 + 1)^2 > 0$
As curve is concave up on either side of $x = -1$, then not a point of inflection.</td><td>$f'''(x) = -24 - 24x$
$f'''(-1) = 0$
Since both second and third derivatives are zero at $x = -1$ then not a point of inflection.</td></tr>
<tr><th colspan="2">Specific behaviours</th></tr>
<tr><td colspan="2">✓ states no, with reasonable attempt to justify
✓ solves $f''(x) = 0$ or locates x value where $f''(x) = 0$
✓ checks concavity either side of point, uses third derivative test, or other valid reasoning</td></tr>
</table>

Question 3 **(9 marks)**

The function f is defined for $x > 0$ by $f(x) = \dfrac{e^{3x-2}}{x}$, and $f''(x) = \dfrac{(9x^2 - 6x + 2)e^{3x-2}}{x^3}$.

(a) Determine the coordinates and nature of all stationary points of the graph of $y = f(x)$. Justify your answer. (6 marks)

Solution
$$f'(x) = \frac{(3e^{3x-2})(x) - (1)(e^{3x-2})}{x^2}$$ $$f'(x) = 0 \to e^{3x-2}(3x-1) = 0 \to x = {}^1/_3$$ $$f''({}^1/_3) = \frac{(1-2+2)e^{-1}}{({}^1/_3)^3} = \frac{27}{e}$$ $$f''({}^1/_3) > 0 \to \text{stationary point is a minimum}$$ $$f({}^1/_3) = \frac{e^{-1}}{{}^1/_3} = \frac{3}{e}$$ The only stationary point of the graph is a minimum at $({}^1/_3, {}^3/_e)$.
Specific behaviours
✓ attempts to use quotient rule ✓ correctly obtains $f'(x)$ ✓ uses $f(x) = 0$ to determine x-coordinate of stationary point ✓ identifies sign of second derivative at stationary point or uses first derivative sign test ✓ correctly identifies nature of stationary point ✓ correct coordinates of stationary point

(b) Show that the graph of $y = f(x)$ has no points of inflection. (3 marks)

Solution
For a point of inflection to exist, $f''(x) = 0 \to 9x^2 - 6x + 2 = 0$. $$e^{3x-2} \neq 0$$ But for this quadratic, $b^2 - 4ac = (-6)^2 - 4(9)(2) = -36$, and so this equation has no solutions as the discriminant is less than zero. Thus, the graph has no points of inflection.
Specific behaviours
✓ uses $f''(x) = 0$ to obtain quadratic ✓ states $e^{3x-2} \neq 0$ ✓ uses quadratic to explain why no points of inflection **Note: may have to change, see how students did the question**

Question 4 **(10 marks)**

The discrete random variable X has a probability function with Var(X) = $\frac{14}{9}$.

$$P(X = x) = \begin{cases} \frac{x}{k}, & x = 1, 2, 3, 4, 5 \\ 0, & \text{otherwise} \end{cases}$$

(a) Show that k = 15. (2 marks)

Solution
$1 = \frac{1}{k} + \frac{2}{k} + \frac{3}{k} + \frac{4}{k} + \frac{5}{k} \qquad \frac{15}{k} = 1 \qquad \therefore k = 15$
Specific behaviours
✓ substitutes values of x and sums ✓ Solves equation

Determine:

(b) (i) P(X < 4 | X > 1) (2 marks)

Solution
$\frac{P(2 \le X \le 3)}{P(X > 1)} = \frac{\frac{2}{15} + \frac{3}{15}}{\frac{14}{15}} = \frac{5}{14}$
Specific behaviours
✓ correct numerator ✓ correct denominator

(ii) E(X) (2 marks)

Solution
$E(X) = \left(\frac{1}{15} + \frac{4}{15} + \frac{9}{15} + \frac{16}{15} + \frac{25}{15}\right)$ $E(X) = \frac{55}{15} = \frac{11}{3}$
Specific behaviours
✓ correct equation ✓ correct E(X) – must be simplified

(c) A second discrete random variable Y is defined to be $Y = aX + b$.

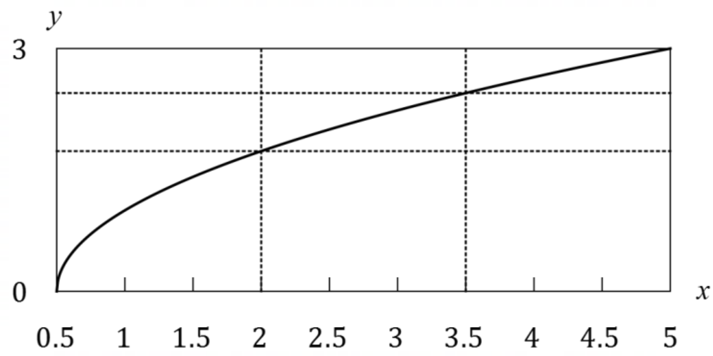
If E(Y) = 2 and the standard deviation of Y is $\sqrt{14}$, determine a and b. (4 marks)

Solution
$E(Y) = aE(X) + b \rightarrow 2 = \frac{11}{3}a + b$ and $S_Y = \lvert a \rvert S_X \rightarrow \sqrt{14} = \frac{\sqrt{14}}{3}\lvert a \rvert$ $\therefore a = 3$ and $b = -9$ or $a = -3$ and $b = 13$
Specific behaviours
✓ correct equation for E(Y) ✓ correct equation to find |a| ✓ one correct value of a and b ✓ both correct values of a and b

Question 5 **(8 marks)**

The graph of $y = \sqrt{2x-1}$ between $x = 0.5$ and $x = 5$ is shown at right.

Approximate values for $\sqrt{3}$ and $\sqrt{6}$ are 1.73 and 2.45 respectively.



(a) Use the areas of the rectangles shown to explain why $6.27 < \int_{0.5}^{5} \sqrt{2x-1}\,dx < 10.77$. (3 marks)

Solution
The value of the integral is the area under the curve between 0.5 and 5. The area of the inscribed rectangles is $\frac{3}{2}(0 + 1.73 + 2.45) = 6.27$, an underestimate. The area of the circumscribed rectangles is $\frac{3}{2}(1.73 + 2.45 + 3) = 10.77$, an overestimate. Hence the value of the integral must lie between these two.
Specific behaviours
✓ derives area approximation using inscribed rectangles ✓ derives area approximation using circumscribed rectangles ✓ explains inequality

(b) Evaluate $\int_{0.5}^{5} \sqrt{2x-1}\,dx$. (3 marks)

Solution
$\int_{0.5}^{5} (2x-1)^{\frac{1}{2}}\,dx = \left[\frac{1}{3}(2x-1)^{\frac{3}{2}}\right]_{0.5}^{5}$ $= \frac{1}{3}(9)^{\frac{3}{2}} - \frac{1}{3}(0)^{\frac{3}{2}}$ $= 9$
Specific behaviours
✓ obtains $(2x-1)^{1.5}$ term in antiderivative ✓ obtains correct antiderivative ✓ substitutes both bounds and simplifies

(c) Evaluate $\int_{0.5}^{5} \left(\sqrt{2x-1} - 3\right) dx$. (2 marks)

Solution
$\int_{0.5}^{5} \left(\sqrt{2x-1} - 3\right) dx = \int_{0.5}^{5} \sqrt{2x-1}\,dx - \int_{0.5}^{5} 3\,dx$ $= 9 - 4.5 \times 3 = -4.5$
Specific behaviours
✓ uses linearity ✓ correct value

Solution
$\int_{0.5}^{5} \left(\sqrt{2x-1} - 3\right) dx = \int_{0.5}^{5} \sqrt{2x-1}\,dx - \int_{0.5}^{5} 3\,dx$ $= 9 - [3x]_{0.5}^{5}$ $= -4.5$
Specific behaviours
✓ obtains correct antiderivative ✓ correct value

Question 6 **(9 marks)**

Let $f(x) = e^{-3x}(\cos 3x + \sin 3x)$.

(a) Determine $f'(x)$, simplifying your answer. (3 marks)

Solution
$f'(x) = (-3e^{-3x})(\cos 3x + \sin 3x) + (e^{-3x})(-3\sin 3x + 3\cos 3x)$ $= -6e^{-3x}\sin 3x$
Specific behaviours
✓ correctly applies product rule   ✓ correctly differentiates trig terms   ✓ simplifies to obtain correct derivative

(b) Hence, show that

$$\int \left(e^{-3x}\sin 3x\right) dx = -\frac{1}{6}e^{-3x}(\cos 3x + \sin 3x) + c,$$

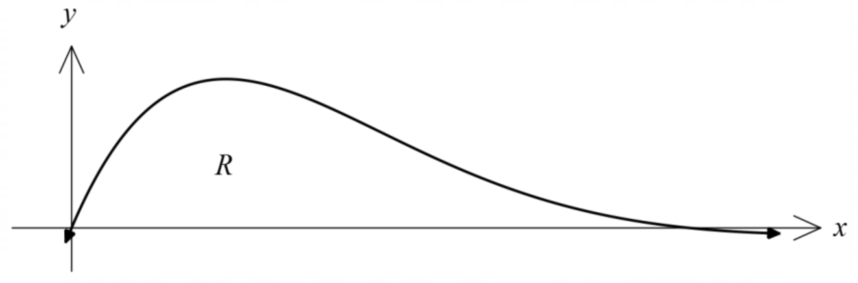
where c is a constant. (3 marks)

Solution
Derivative of LHS (using derivative of integral of a function is original function): $\frac{d}{dx}\left(\int \left(e^{-3x}\sin 3x\right) dx\right) = e^{-3x}\sin 3x$ Derivative of RHS (using part (a)): $\frac{d}{dx}\left(-\frac{1}{6}e^{-3x}(\cos 3x + \sin 3x) + c\right) = -\frac{1}{6} \times (-6e^{-3x}\sin 3x) = e^{-3x}\sin 3x$ Hence LHS=RHS.
Specific behaviours
✓ differentiates LHS   ✓ differentiates RHS and   ✓simplifies to equal derivative of LHS

Solution
$\frac{d}{dx}[e^{-3x}(\cos 3x + \sin 3x)] = -6e^{-3x}\sin 3x$ $\int \frac{d}{dx}[e^{-3x}(\cos 3x + \sin 3x)] = \int -6e^{-3x}\sin 3x\, dx$ $e^{-3x}(\cos 3x + \sin 3x) + k = \int -6e^{-3x}\sin 3x\, dx$ $\therefore \int (e^{-3x}\sin 3x)\, dx = -\frac{1}{6}e^{-3x}(\cos 3x + \sin 3x) + c$
Specific behaviours
✓ integrates all terms of result from (a)   ✓ uses fundamental theorem to simplify LHS   ✓ obtains required result, with constant

(c) The graph of $y = e^{-3x} \sin 3x$ is shown below. Determine the area of the region R, bounded by the curve and the x-axis.

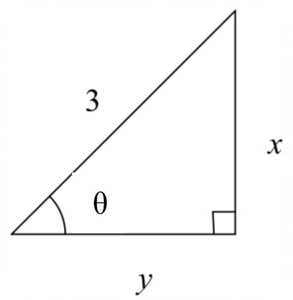
(3 marks)



Solution
$\sin 3x = 0 \Rightarrow 3x = \pi, x = \frac{\pi}{3}$ $\int_0^{\frac{\pi}{3}} e^{-3x} \sin 3x \, dx = \left[\frac{-e^{-3x}}{6}(\cos 3x + \sin 3x)\right]_0^{\frac{\pi}{3}}$ $= \left(-\frac{e^{-\pi}}{6}(\cos \pi + \sin \pi)\right) - \left(-\frac{e^0}{6}(\cos 0 + \sin 0)\right)$ $= \frac{e^{-\pi}}{6} + \frac{1}{6} = \frac{e^{-\pi} + 1}{6}$
Specific behaviours
✓ forms integral with correct bounds   ✓ writes antiderivative and substitutes bounds   ✓ simplifies

Question 7 **(6 marks)**

Given $\cos(2x) = \cos^2 x - \sin^2 x$ and the diagram below;



(a) show that the area of the triangle is given by $A(\theta) = \frac{9}{2}\sin\theta\cos\theta$. (2 marks)

Solution
$x = 3\sin\theta \qquad y = 3\cos\theta$ $A = \frac{1}{2}ab$ $A = \frac{1}{2}(3\cos\theta\, 3\sin\theta)$ $A = \frac{9}{2}(\cos\theta\sin\theta)$
Specific behaviours
✓ expresses x and y in terms of sin and cos ✓ substitutes into area formula and simplifies to get required equation

(b) Determine the value of θ that will maximise the area of the triangle. (4 marks)

Solution
Derivative of A with respect to θ: $$A'(\theta) = \frac{9}{2}(\sin\theta\,(-\sin\theta) + \cos\theta\cos\theta) = \frac{9}{2}(\cos^2\theta - \sin^2\theta) = \frac{9}{2}(\cos 2\theta)$$ Derivative will be zero when $A'(\theta) = 0$. Hence $$0 = \frac{9}{2}(\cos 2\theta)$$ $$0 = \cos 2\theta$$ $$\frac{\pi}{2} = 2\theta \implies \frac{\pi}{4} = \theta$$ Check using second derivative: $$A''(\theta) = -\frac{9}{2}\sin(2\theta)\cdot 2 = (-9(\sin 2\theta) \quad \Rightarrow A''\left(\frac{\pi}{4}\right) = -9 \ \therefore Max$$ Area will be a maximum when $\theta = \frac{\pi}{4}$
Specific behaviours
✓ obtains derivative   ✓ equates derivative to zero and finds correct value for θ ✓ uses second derivative to check that it is a maximum for θ ✓ states that area will be a maximum when $\theta = \frac{\pi}{4}$

End of questions

Supplementary page

Question number: _________

Supplementary page

Question number: _________